Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 68 (2004) 827–838

FIRST-APPROXIMATION INSTABILITY CRITERIA FOR NON-STATIONARY LINEARIZATIONS†

G. A. LEONOV
St Petersburg
e-mail: leonov@math.spbu.ru

(*Received* 29 *May* 2003)

Perron effect of sign inversion of characteristic Lyapunov exponents for solving non-linear systems and their first approximations under the same initial conditions are considered. The Krasovskii and Lyapunov instability criteria are proved using the Perron–Vinograd triangulation method.

The current problem of substantiating non-stationary linearizations for complex, non-periodic motions is remarkably similar to the situation 120 years ago. The founders of the theory of automatic control, Maxwell (1868) and Vyshnegradskii (1876) [1], boldly carried out linearization in the neighbourhood of steady motions, leaving the substantiation of such linearization to Poincaré (1886) and Lyapunov (1892). Among a wide circle of chaotic dynamics specialists, the firm conviction arose that the positiveness of the highest characteristic exponent of a first-approximation linear system implies instability of the solutions of the initial system (see, for example, [2, p. 227; 3, p. 72; 4, p. 26; 5, p. 323; 6]). Moreover, there have been a vast number of computer experiments where different numerical procedures have been used to calculate the characteristic exponents and Lyapunov exponents of first-approximation linear systems. Here, the authors, in general, entirely ignoring the substantiation of the linearization procedure, have constructed, from the numerical values of the characteristic exponents obtained, different numerical characteristic of attractors of the initial non-linear systems (Lyapunov dimensionalities, metric entropies, etc.)

Occasionally, partial substantiation of a linearization procedure is deduced using computer experiments. For example, computer experiments [2, 7] show agreement between the Lyapunov and Hausdorff dimensionality of Xenon, Kaplan–York and Zaslavskii attractors. However, for Xenon and Lyapunov B-attractors there is no such agreement [8, 9].

This paper is devoted to the problem of substantiating a linearization procedure where a first-approximation system has a positive characteristic exponent. It is shown that, in this case, Perron effects of sign inversion of characteristic exponents of the solutions of the initial system and of the first-approximation system under the same initial conditions can be observed; these effects indicate the difficulties that must be overcome when substantiating the linear procedure. The Krasovskii and Lyapunov instability criteria are proved using the Perron–Vinograd triangulation method.

1. PERRON EFFECTS

Perron [10] discovered the effect of sign inversion of characteristic exponents for the solutions of special classes of non-linear systems and their first approximations. He constructed a non-linear system, the first approximation of which had negative characteristic exponents, while nearly all its solutions possessed positive characteristic exponents.

Here we will examine the analogous effect of a change in the sign of characteristic exponents "the other way round": the solution of the first-approximation system has a positive characteristic exponent while the solution of the initial system with the same initial data has a negative characteristic exponent.

†*Prikl. Mat. Mekh.* Vol. 68, No. 6, pp. 925–937, 2004.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2004.11.004

Consider the system

$$\begin{aligned}\dot{x}_1 &= [\sin\ln(t+1)+\cos\ln(t+1)-2a]x_1 + x_3 - x_2^2\\ \dot{x}_2 &= -ax_3, \quad \dot{x}_3 = -2ax_3\end{aligned} \tag{1.1}$$

on an invariant manifold

$$M = \{x_1 \in R^1, x_2^2 = x_3\}$$

The number a satisfies the condition

$$1 < 2a < 1 + \exp(-\pi)/2$$

The solution of system (1.1) on the manifold M have the form

$$\begin{aligned}x_1(t) &= \exp[(t+1)\sin\ln(t+1) - 2at]x_1(0)\\ x_2(t) &= \exp[-at]x_2(0), \quad x_3(t) = \exp[-2at]x_3(0); \quad x_3(0) = x_2(0)^2\end{aligned} \tag{1.2}$$

It is obvious that these solutions have negative Lyapunov exponents.
The solution of the first-approximation system

$$\begin{aligned}\dot{z}_1 &= [\sin\ln(t+1)+\cos\ln(t+1)-2a]z_1 + z_3\\ \dot{z}_2 &= -az_2, \quad \dot{z}_3 = -2az_3\end{aligned} \tag{1.3}$$

on the manifold M have the form

$$z_1(t) = \exp[(t+1)\sin\ln(t+1) - 2at]\times$$

$$\times\left[z_1(0) + z_3(0)\int_0^t \exp[-\tau + 1]\sin\ln(\tau+1)\Big]d\tau\right]$$

$$z_2(t) = \exp[-at]z_2(0), \quad z_3(t) = \exp[-2at]z_3(0); \quad z_3(0) = z_2(0)^2$$

Assuming that $t = \exp[(2n + 1/2)\pi] - 1$, where n is an integer, we obtain the estimate [11, p. 369]

$$\int_0^t \exp[-(\tau+1)\sin\ln(\tau+1)]d\tau >$$

$$> \exp\left[\frac{1}{2}(t+1)\exp(-\pi)\right](t+1)\left[\exp\left(-\frac{2\pi}{3}\right) - \exp(-\pi)\right]$$

Therefore, for the given values of t, we have the inequality

$$\exp[(t+1)\sin\ln(t+1) - 2at]\int_0^t \exp[-(\tau+1)\sin\ln(\tau+1)]d\tau >$$

$$> \exp\left[\frac{1}{2}(2+\exp(-\pi))\right]\left[\exp\left(-\frac{2\pi}{3}\right) - \exp(-\pi)\right]\exp\left[\left(1 - 2a + \frac{1}{2}\exp(-\pi)\right)t\right]$$

Hence, on the manifold M when $z_3(0) \neq 0$, the following inequality holds

$$\overline{\lim_{t\to+\infty}}\frac{1}{t}|\ln|z_1(t)|| > 0$$

Note that, for any solutions of systems (1.1) and (1.3), we have the relations

$$(x_2(t)^2 - x_3(t))^{\bullet} = -2a(x_2(t)^2 - x_3(t))$$

$$(z_2(t)^2 - z_3(t))^{\bullet} = -2a(z_2(t)^2 - z_3(t))$$

Therefore

$$x_2(t)^2 - x_3(t) = \exp[-2at][x_2(0)^2 - x_3(0)]$$

$$z_2(t)^2 - z_3(t) = \exp[-2at][z_2(0)^2 - z_3(0)]$$

Hence we conclude that the manifold M is an exponentially attracting invariant set for systems (1.1) and (1.3). This means that, from the relation $x_2(0)^2 = x_3(0)$, we have the equality $x_2(t)^2 = x_3(t)$ for all $t \in R^1$, and that for any initial conditions

$$|x_2(t)^2 - x_3(t)| \le \exp[-2at]|x_2(0)^2 - x_3(0)|$$

Thus, systems (1.1) and (1.3) have the same invariants exponentially attracting manifold M on which almost all solutions of the first-approximation system (1.3) have a positive characteristic exponent, while all solutions of the initial system (1.1) have negative characteristic exponents.

Here, the Perron effect is observed on the entire manifold

$$\{x_1 \in R^1,\ x_3 = x_2^2 \neq 0\}$$

In order to construct an exponentially stable system whose first approximation has a positive characteristic exponent, we will change system (1.1) in the following way

$$\begin{aligned} \dot{x}_1 &= [\sin\ln(t+1) + \cos\ln(t+1) - 2a]x_1 + x_3 - x_2^2 \\ \dot{x}_2 &= F(x_2, x_3), \quad \dot{x}_3 = G(x_2, x_3) \end{aligned} \tag{1.4}$$

The function $F(x_2, x_3)$ and $G(x_2, x_3)$ have the form

$$F(x_2, x_3) = \pm 2x_3 - ax_2, \quad G(x_2, x_3) = \mp x_2 - \varphi(x_2, x_3)$$

where the upper sign is taken when $x_2 > 0, x_3 > x_2^2$ and when $x_2 < 0, x_3 < x_2^2$, and the lower sign is taken when $x_2 > 0, x_3 < x_2^2$ and when $x_2 < 0, x_3 > x_2^2$. The function $\varphi(x_2, x_3)$ is defined as following:

$$\varphi(x_2, x_3) = \begin{cases} 4ax_3 & \text{when } x_3 > 2x_2^2 \text{ and when } x_3 < -2x_2^2 \\ 2ax_3 & \text{when } -2x_2^2 < x_3 < 2x_2^2 \end{cases}$$

The solutions of system (1.4) are understood in Filippov's sense [12]. According to this definition for the function F and G, the system

$$\dot{x}_2 = F(x_2, x_3), \quad \dot{x}_3 = G(x_2, x_3) \tag{1.5}$$

on lines of discontinuity $\{x_2 = 0\}$ and $\{x_3 = x_2^2\}$ has sliding solutions, which are described by the equations

$$x_2(t) \equiv 0, \quad \dot{x}_3(t) = -4ax_3(t)$$

and

$$\dot{x}_2(t) = -ax_2(t), \quad \dot{x}_3(t) = -2ax_3(t), \quad x_3(t) \equiv x_2(t)^2$$

Here, the solutions of system (1.5) with the initial data $x_2(0) \neq 0$, $x_3(0) \in R^1$ fall on the curve $\{x_3 = x_2^2\}$ in a finite time not exceeding 2π. The phase portrait of such a system is shown in Fig. 1.

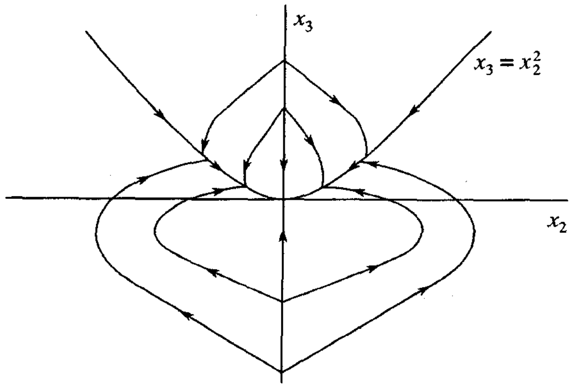

Fig. 1

From the considerations given here it follows that, for the solutions of system (1.4) with the initial data

$$x_1(0) \in R^1, \quad x_2(0) \neq 0, \quad x_3(0) \in R^1$$

at $t \geq 2\pi$, the inequalities

$$F(x_2(t), x_3(t)) = -ax_2(t), \quad G(x_2(t), x_3(t)) = -2ax_3(t)$$

occur. Therefore, in such solutions, system (1.3) is a first-approximation system when $t \geq 2\pi$.

As shown above, this system has a positive characteristic exponent. At the same time, all solutions of system (1.4) tend to zero exponentially.

2. INSTABILITY CRITERIA

Consider the system

$$dx/dt = A(t)x + f(t, x), \quad t \geq 0, \quad x \in R^n \tag{2.1}$$

where $A(t)$ is continuous $n \times n$ matrix bounded in $[0, \infty]$. We will assume that the vector function $f(t, x)$ is continuous and, in a certain neighbourhood $\Omega(0)$ of the point $x = 0$, the inequality

$$|f(t, x)| \leq \kappa |x|^\nu, \quad \forall t \geq 0, \quad \forall x \in \Omega(0) \tag{2.2}$$

holds. The number κ and ν are such that $\kappa > 0$ and $\nu > 1$.

We recall here the definition of Lyapunov and Krasovskii stability.

Definition 1. The solution $x(t) \equiv 0$ of system (2.1) is termed Lyapunov stable if, for any $\varepsilon > 0$ and $t_0 \geq 0$, a number $\delta(\varepsilon, t_0)$ exists such that, for the solution $x(t, t_0, x_0)$ satisfying the condition $|x_0| \leq \delta(\varepsilon, t_0)$, the inequality

$$|x(t, t_0, x_0)| \leq \varepsilon, \quad \forall t \geq t_0$$

is satisfied.

Definition 2. The solution $x(t) \equiv 0$ of system (2.1) is termed Krasovskii stable if positive numbers $\delta(t_0)$ and $R(t_0)$ exist such that, for solutions $x(t, t_0, x_0)$ satisfying the condition $|x_0| \leq \delta(t_0)$, the inequality

$$|x(t, t_0, x_0)| \leq R(t_0)|x_0|, \quad \forall t \geq t_0$$

is satisfied.

We recall that, by definition, Lyapunov (Krasovskii) instability is a negation of the corresponding type of stability.

We will introduce into consideration the normal fundamental matrix

$$Z(t) = (z_1(t), \ldots, z_n(t)) \tag{2.3}$$

consisting of linearly independent solutions $z_j(t)$ of the first-approximation linear system

$$dz/dt = A(t)z \tag{2.4}$$

To analysis the instability, we will use the Perron–Vinograd method of triangulation of a linear system [10, 13].

We will carry out the Schmidt orthogonalization procedure for the solution $z_j(t)$ forming matrix (2.3)

$$\begin{aligned} v_1(t) &= z_1(t) \\ v_2(t) &= z_2(t) - v_1(t)^* z_2(t)\frac{v_1(t)}{|v_1(t)|^2} \\ &\dots\dots\dots\dots \\ v_n(t) &= z_n(t) - v_1(t)^* z_n(t)\frac{v_1(t)}{|v_1(t)|^2} - \dots - v_{n-1}(t)^* z_n(t)\frac{v_{n-1}(t)}{|v_{n-1}(t)|^2} \end{aligned} \tag{2.5}$$

The asterisk denotes transposition.

The equalities

$$v_i(t)^* v_j(t) = 0, \quad \forall j \neq i; \quad |v_j(t)|^2 = v_j(t)^* z_j(t) \tag{2.6}$$

follow from relations (2.5). The following assertion follow from the latter equality.

Lemma 1. The following estimate holds

$$|v_j(t)| \le |z_j(t)|, \quad \forall t \ge 0 \tag{2.7}$$

The answer to the question as to how greatly the vector function $v_j(t)$ may decrease compared with the initial system of vector $z_j(t)$ gives the following assertion.

Lemma 2. If for a certain number C the inequality

$$\prod_{j=1}^{n} |z_j(t)| \le C \exp\int_0^t \operatorname{tr} A(s)ds, \quad \forall t \ge 0 \tag{2.8}$$

if satisfied, then a number $r > 0$ exists for which the following estimate holds

$$|z_j(t)| \le r|v_j(t)|, \quad \forall t \ge 0, \quad j = 1, \dots, n \tag{2.9}$$

Proof. We will introduce into consideration the matrix

$$\tilde{Z}(t) = \left(\frac{z_1(t)}{|z_1(t)|}, \dots, \frac{z_n(t)}{|z_n(t)|}\right)$$

From the Ostrogradskii–Liouville formula [13] and inequality (2.8) we have the relation

$$\begin{aligned} |\det \tilde{Z}(t)| &= |\det(z_1(0), \dots, z_n(0))|(|z_1(t)|, \dots, |z_n(t)|)^{-1} \exp\int_0^t \operatorname{tr} A(s)ds \ge \\ &\ge C^{-1}|\det(z_1(0), \dots, z_n(0))|, \ \forall t \ge 0 \end{aligned}$$

From this it follows that, for the linear subspace $L(t)$ drawn on the vectors $z_1(t), \dots, z_m(t)$ $(m < n)$, a number $\varepsilon \in (0,1)$ will be found such that the estimate

$$\frac{|z_{m+1}(t)^* e(t)|}{|z_{m+1}(t)|} \le 1 - \varepsilon, \quad \forall t \ge 0 \tag{2.10}$$

holds for all $e(t) \in L(t)$ satisfying the equality $|e(t)| = 1$.

Relations (2.5) when $j > 1$ can be rewritten as follows:

$$\frac{v_j(t)}{|z_j(t)|} = \prod_{i=1}^{j-1}\left(I - \frac{v_i(t)v_i(t)^*}{|v_i(t)|^2}\right)\frac{z_j(t)}{|z_j(t)|} \tag{2.11}$$

where I is the identity matrix.

We will now assume that the assertion of the lemma does not occur. In this case a sequence $t_k \to +\infty$ exists for which

$$\lim_{k\to+\infty}\frac{v_j(t_k)}{|z_j(t_k)|} = 0$$

But then, from equality (2.11) we obtain that a number $l < j$ exist for which

$$\lim_{k\to+\infty}\left[\frac{z_j(t_k)}{|z_j(t_k)|} - \frac{v_l(t_k)}{|v_l(t_k)|}\right] = 0 \tag{2.12}$$

Since $v_l(t) \in L(t)$, relation (2.10) and (2.12) are contradictory. The contradiction obtained proves estimate (2.9).

From the reasoning set out here it can be seen that condition (2.8) is a necessary and sufficient condition for a number $r > 0$ to exist for which estimate (2.9) is satisfied.

Note that condition (2.8) is the necessary and sufficient condition for non-degeneracy when $t \to +\infty$ of the normalized fundamental matrix of the first-approximation system (2.4)

$$\underline{\lim}_{t\to+\infty}|\det\tilde{Z}(t)| > 0$$

We will now describe the Perron–Vinograd triangulation procedure.

We will introduce the unitary matrix.

$$U(t) = \left(\frac{v_1(t)}{|v_1(t)|}, \ldots, \frac{v_n(t)}{|v_n(t)|}\right)$$

and make the following change in system (2.4): $z = U(t)w$.

From the unitary nature of the matrix $U(t)$ it follows that for the columns $w_j(t)$ of the matrix

$$W(t) = (w_1(t), \ldots, w_n(t)) = U(t)^*Z(t)$$

the relations $|w_j(t)| = |z_j(t)|$ are satisfied.

From relations (2.5) and (2.6) it follows that the matrix $W(t)$ has the following triangular form

$$W(t) = \begin{pmatrix} |v_1(t)| & \ldots & \\ & \ddots & \vdots \\ 0 & & |v_n(t)| \end{pmatrix} \tag{2.13}$$

The matrix $W(t)$ is the fundamental matrix of the system

$$dw/dt = B(t)w \tag{2.14}$$

where

$$B(t) = U(t)^{-1}A(t)U(t) - U(t)^{-1}\dot{U}(t) \tag{2.15}$$

From the fact that $W(t)$ is a higher triangular matrix it follows that $W(t)^{-1}$ and $\dot{W}(t)$ are also higher triangular matrices. Therefore, the matrix

$$B(t) = \dot{W}(t)W(t)^{-1}$$

is a higher triangular matrix of the form

$$B(t) = \begin{pmatrix} (\ln|v_1(t)|)^{\bullet} & \dots & \\ & \ddots & \vdots \\ 0 & & (\ln|v_n(t)|)^{\bullet} \end{pmatrix} \tag{2.16}$$

We will show that, if the matrix $A(t)$ is bounded when $t \geq 0$, then the matrices $B(t)$, $U(t)$ and $\dot{U}(t)$ are also bounded when $t \geq 0$. Boundedness of the matrix $U(t)$ always occurs and is obvious. Therefore, the matrix

$$U(t)^{-1}A(t)U(t) = U(t)^*A(t)U(t)$$

is also bounded.

From the unitary nature of the matrix $U(t)$ we have the identity

$$(U(t)^{-1}\dot{U}(t))^* = -U(t)^{-1}\dot{U}(t) \tag{2.17}$$

From this, and from relations (2.15) and (2.16), it follows that the modulus of an element of the matrix $U(t)^{-1}\dot{U}(t)$ is identical with the modulus of a certain element of the matrix $U(t)^{-1}A(t)U(t)$. Thus, the matrix $U(t)^{-1}\dot{U}(t)$ is bounded when $t \geq 0$. The boundedness of the matrix $B(t)$ follows from this and from equality (2.15). The boundedness of $\dot{U}(t)$ follows from the boundedness of $B(t)$ and the equality $\dot{U}(t) = A(t)U(t) - U(t)B(t)$.

We will now prove one further useful estimate for the vector function $v_n(t)$.

Lemma 3. The following estimate holds

$$\frac{|v_n(t)|}{|v_n(\tau)|} \geq \exp\left[\int_\tau^t \operatorname{tr} A(s)ds\right] \prod_{j=1}^{n-1} \frac{|v_j(\tau)|}{|z_j(t)|} \tag{2.18}$$

Proof. From relation (2.13) we have the equality

$$\frac{|v_n(t)|}{|v_n(\tau)|} = \frac{\det W(t) \prod_{j=1}^{n-1} |v_j(\tau)|}{\det W(\tau) \prod_{j=1}^{n-1} |v_j(t)|}$$

From the Ostrogradskii–Liouville formula and relations (2.14), (2.15) and (2.17) we have the equalities

$$\det W(t) = \det W(\tau) \exp \int_\tau^t \operatorname{tr} B(s)ds = \det W(\tau) \int_\tau^t \operatorname{tr} A(s)ds$$

The lemma follows immediately from these equalities and estimate (2.7).

The triangulation method described above and Lemma 3 enable the following result to be made almost obvious.

Theorem 1. If the inequality

$$\sup_k \varlimsup_{t \to +\infty} \left[\frac{1}{t}\left(\int_0^t \operatorname{tr} A(s)ds - \sum_{j \neq k} \ln|z_j(t)|\right)\right] > 0 \tag{2.19}$$

is satisfied, the solution $x(t) \equiv 0$ of system (2.1) is Krasovskii unstable.

Proof. Without loss of generality, it can be assumed that in condition (2.19) the supremum with respect to k is attained when $k = n$.

We will make the change $x = U(t)y$ in system (2.1)

$$dy/dt = B(t)y + g(t, y); \quad g(t, y) = U(t)^{-1}f(t, U(t)y) \tag{2.20}$$

The matrix $B(t)$ is determined from formula (2.15).

Thus, the final equation of system (2.20) will take the form

$$\dot{y}_n = (\ln|\upsilon_n(t)|)^{\bullet} y_n + g_n(t, y) \tag{2.21}$$

Here, y_n and g_n are the nth components of the vectors y and g.

We will now assume that the solution $x(t) \equiv 0$ is Krasovskii stable. This means the existence for a certain neighbourhood $x = 0$ of a number $R > 0$ such that the estimate

$$|x(t, x_0)| \le R|x_0|, \quad \forall t \ge 0 \tag{2.22}$$

is satisfied. Here, $x(0, x_0) = x_0$.

From conditions (2.2) and (2.22) we have the estimate

$$|g(t, y(t))| \le \kappa R^{\nu}|y(0)|^{\nu} \tag{2.23}$$

From conditions (2.19), according to Lemma 3, we will obtain the existence of a number $\mu > 0$ such that, for sufficiently high values of t, the following estimate holds

$$\ln|\upsilon_n(t)| \ge \mu t \tag{2.24}$$

Note that the solution $y_n(t)$ of Eq. (2.21) can be represented in the form

$$y_n(t) = \frac{|\upsilon_n(t)|}{|\upsilon_n(0)|}\left(y_n(0) + \int_0^t \frac{|\upsilon_n(0)|}{|\upsilon_n(s)|} g(s, y(s))ds\right) \tag{2.25}$$

From estimate (2.24) it follows that a number $\rho > 0$ exists such that the inequality

$$\int_0^t \frac{|\upsilon_n(0)|}{|\upsilon_n(s)|} ds \le \rho, \quad \forall t \ge 0 \tag{2.26}$$

is satisfied.

We will now take the initial conditions $x_0 = U(0)y(0)$ such that $y_n(0) = |y(0)| = \delta$, where $\delta > \rho\kappa R^{\nu}\delta^{\nu}$. In this case, from relations (2.23) to (2.26) we obtain for sufficiently high $t \ge 0$, the estimate

$$y_n(t) \ge \exp(\mu t)(\delta - \rho\kappa R^{\nu}\delta^{\nu})$$

From this we immediately have the relation

$$\varliminf_{t \to +\infty} y_n(t) = +\infty$$

This relation contradicts the assumption concerning the Krasovskii stability of the trivial solution of system (2.1). The theorem is proved.

Remark regarding the method for proving Theorem 1. If the presence of Lyapunov stability were assumed and an attempt were made to derive a contradiction to this, just as was done in the proof in relation to Krasovskii stability, then in this case it would be necessary to prove the inequality

$$y_n(0) + \int_0^{+\infty} \frac{|\upsilon_n(0)|}{|\upsilon_n(s)|} g(s, y(s))ds \ne 0 \tag{2.27}$$

This inequality is easy to ensure when it is a matter of Krasovskii stability, and the need to prove it is a difficult obstacle to overcome when considering Lyapunov stability.

An analogous scheme for reducing the problem to a single scalar equation of type (2.21) was used by Chetayev [14, 15] when obtaining instability criteria. A similar difficulty in proving inequality (2.27) also exists in the scheme proposed by Chetayev. Therefore, the Chetayev method now enables the Krasovskii instability criteria to be obtained.

The procedure for obtaining Lyapunov instability criteria requires further development. Such a development with certain additional limitations will be given in Theorem 2.

Condition (2.19) of Theorem 1 is satisfied if the inequality

$$\Lambda - \Gamma > 0 \tag{2.28}$$

holds. Here, Λ is the maximum characteristic exponent and Γ is the incorrectness coefficient [13]. We recall that

$$\Gamma = \Sigma - \varliminf_{t \to +\infty} \frac{1}{t}\int_0^t \operatorname{tr} A(s) ds$$

The condition for Krasovskii instability (2.28) was obtained by Chetayev [14, 15] with a stronger requirement for the analyticity of the function $f(t, x)$.

Note also that, for system (1.3), $\Gamma = \Lambda + 2a + 1$. Therefore, condition (2.28) for the systems examined in Section 1 is not satisfied.

Theorem 2. If, for certain numbers $C > 0$, $\beta > 0$, $\alpha_j < \beta$ $(j = 1, \ldots, n-1)$ the conditions

$$\prod_{j=1}^{n} |z_j(t)| \le C \exp \int_0^t \operatorname{tr} A(s) ds, \quad \forall t \ge 0 \quad \text{when} \quad n > 2 \tag{2.29}$$

$$|z_j(t)| \le C\exp(\alpha_j(t-\tau))|z_j(\tau)|, \quad \forall t \ge \tau \ge 0, \quad \forall j = 1, \ldots, n-1 \tag{2.30}$$

$$\frac{1}{(t-\tau)}\int_\tau^t \operatorname{tr} A(s) ds > \beta + \sum_{j=1}^{n-1} \alpha_j, \quad \forall t \ge \tau \ge 0 \tag{2.31}$$

are satisfied, then the zero solution of system (2.1) is Lyapunov unstable.

Proof. We recall that the fundamental matrix $W(t)$ of system (2.14) has the form of (2.13). The column vectors of this matrix $w_j(t)$ $(j = 1, \ldots, n-1)$ have the form

$$w_1(t) = \begin{pmatrix} w_{11}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, w_{n-1}(t) = \begin{pmatrix} w_{n-1,1}(t) \\ \vdots \\ w_{n-1,n-1}(t) \\ 0 \end{pmatrix}$$

Therefore, the matrix $\tilde{W}(t)$, obtained by cancelling out the final column and the final row of the matrix $W(t)$, is the fundamental matrix of the system

$$\dot{\tilde{w}} = \tilde{B}(t)\tilde{w}, \quad \tilde{w} \in R^{n-1}$$

with the matrix $\tilde{B}(t)$, obtained by cancelling out the final column and the final row of the matrix $B(t)$.

Condition (2.30) and the identities $|w_j(t)| \equiv |z_j(t)|$, which follow on from the unitary nature of the matrix $U(t)$, yield the estimates

$$|\tilde{w}_j(t)| \le C\exp(\alpha_j(t-\tau))|\tilde{w}_j(\tau)|, \quad \forall t \ge \tau \ge 0, \quad \forall j = 1, \ldots, n-1 \tag{2.32}$$

Furthermore, condition (2.29), according to Lemma 2, yields the estimates (2.9), and conditions (2.30) and (2.31), according to Lemma 3, give the estimate

$$\frac{|v_n(t)|}{|v_n(\tau)|} \ge C^{1-n}\exp(\beta(t-\tau))\prod_{j=1}^{n-1}\frac{|v_j(\tau)|}{|z_j(\tau)|}, \quad \forall t \ge \tau \ge 0 \tag{2.33}$$

From relations (2.9) and (2.33) we obtain the inequality

$$\frac{|\upsilon_n(t)|}{|\upsilon_n(\tau)|} \geq (Cr)^{1-n}\exp(\beta(t-\tau)), \quad \forall t \geq \tau \geq 0 \tag{2.34}$$

Since when $n = 2$ we have $\upsilon_1(t) = z_1(t)$, from inequality (2.33) we obtain the estimate

$$\frac{|\upsilon_2(t)|}{|\upsilon_2(\tau)|} \geq C^{-1}\exp(\beta(t-\tau)), \quad \forall t \geq \tau \geq 0$$

without assumption (2.29).

For system (2.1) we now make the replacement

$$x = e^{dt}U(t)y \tag{2.35}$$

Here, we choose the number $d > 0$ such that

$$\alpha < d < \beta$$

where

$$\alpha = \max \alpha_j, \quad j = 1, \ldots, n-1$$

As a result of this replacement, we obtain the system

$$\frac{dy}{dt} = (B(t) - dI)y + g(t, y) \tag{2.36}$$

where

$$g(t, y) = e^{-dt}U(t)^{-1}f(t, e^{dt}U(t)y)$$

From condition (2.2) it follows that, for any number $\rho > 0$, a neighbourhood $\Phi(0)$ of point $y = 0$ exists such that

$$|g(t, y)| \leq \rho|y|, \quad \forall t \geq 0, \quad \forall y \in \Phi(0) \tag{2.37}$$

Note that for the system

$$\dot{\tilde{y}} = (\tilde{B}(t) - dI)\tilde{y}, \quad \tilde{y} \in R^{n-1} \tag{2.38}$$

by virtue of relations (2.32) we have to estimate

$$|\tilde{y}(t)| \leq C\exp[(\alpha - d)(t-\tau)]|\tilde{y}(\tau)|, \quad \forall t \geq \tau$$

Therefore, by Malkin's theorem [11], a continuously differentiable matrix $H(t)$ bounded on $[0, +\infty)$ and positive numbers ρ_1 and ρ_2 exist for which

$$\tilde{y}^*(\dot{H}(t) + 2H(\tilde{B}(t) - dI))\tilde{y} \leq -\rho_1|\tilde{y}|^2, \quad \forall \tilde{y} \in R^{n-1}, \quad \forall t \geq 0 \tag{2.39}$$

$$\tilde{y}^*H(t)\tilde{y} \geq \rho_2|\tilde{y}|^2, \quad \forall \tilde{y} \in R^{n-1}, \quad \forall t \geq 0 \tag{2.40}$$

For the scalar equation

$$\dot{y}_n = [(\ln|\upsilon_n(t)|)^{\cdot} - d]y_n, \quad y_n \in R^1$$

for $n \neq 2$, from relation (2.34) we have the estimate

$$|y_n(t)| \geq (Cr)^{1-n}\exp[(\beta - d)(t-\tau)]|y_n(\tau)|, \quad \forall t \geq \tau \geq 0$$

When $n = 2$, the analogous estimate has the form

$$|y_2(t)| \geq C^{-1}\exp[(\beta - d)(t-\tau)]|y_2(\tau)|, \quad \forall t \geq \tau \geq 0$$

Therefore, by Malkin's theorem [11], a continuously differentiable function $h(t)$ bounded on $[0, +\infty]$ and positive numbers ρ_3 and ρ_4 exist for which

$$\dot{h}(t) + 2h(t)[(\ln|\upsilon_n(t)|)^{\bullet} - d] \leq -\rho_3, \quad h(t) \leq -\rho_4, \quad \forall t \geq 0 \tag{2.41}$$

We will now show that the function

$$V(t, y) = \tilde{y}^* H(t)\tilde{y} + \omega h(t) y_n^2$$

for sufficiently large ω will be a Lyapunov function satisfying, for system (2.36), all the conditions of Lyapunov's classical instability theorem.

Indeed, system (2.36) can be written in the form

$$\begin{aligned} \dot{\tilde{y}} &= (\tilde{B}(t) - dI)\tilde{y} + q(t)y_n + \tilde{g}(t, \tilde{y}, y_n) \\ \dot{y}_n &= ((\ln|\upsilon_n(t)|)^{\bullet} - d)y_n + g_n(t, \tilde{y}, y_n) \end{aligned} \tag{2.42}$$

where $q(t)$ is a certain vector function bounded in $[0,+\infty]$, and $\tilde{g}$ and g_n are such that

$$g(t, y) = \begin{pmatrix} \tilde{g}(t, y) \\ g_n(t, y) \end{pmatrix}$$

Therefore, from estimates (2.39) and (2.41) we have the inequalities

$$\begin{aligned} &\dot{V}(t, y) \leq -\rho_1|\tilde{y}|^2 - \omega\rho_3 y_n^2 + 2\tilde{y}^* H(t) q(t) y_n + \\ &+ 2\tilde{y}^* H(t)\tilde{g}(t, \tilde{y}, y_n) + 2\omega h(t) y_n g_n(t, \tilde{y}, y_n) \leq \\ &\leq -\rho_1|\tilde{y}|^2 - \omega\rho_3 y_n^2 + 2[(|y_n||\tilde{y}|\sup_t|H(t)||q(t)| + \\ &+ |\tilde{y}|\sup_t|H(t)|\rho(|\tilde{y}| + |y_n|) + \omega|y_n|\sup_t|h(t)|\rho(|\tilde{y}| + |y_n|)] \end{aligned}$$

From these inequalities and the boundedness when $t \geq 0$ of the matrix function $H(t)$, the vector function $q(t)$ and the function $h(t)$, it follows that, for sufficiently large ω and sufficiently small ρ, a positive number θ will be found for which

$$\dot{V}(t, y) \leq -\theta|y|^2 \tag{2.43}$$

The boundedness of $H(t)$ and $h(t)$ result in the existence of a number a for which

$$|y|^2 \geq -aV(t, y), \quad \forall t \geq 0, \quad \forall y \in R^n$$

Therefore, from this and from inequality (2.43) we have the inequality

$$\dot{V}(t, y) \leq a\theta V(t, y), \quad \forall t \geq 0, \quad \forall y \in R^n \tag{2.44}$$

We will now take the initial data $y(0)$ such that $V(0, y(0)) < 0$. Then, from inequality (2.43) it follows that

$$V(t, y(t)) < 0, \quad \forall t \geq 0$$

and, by virtue of inequality (2.44), we have the estimate

$$-V(t, y(t)) \geq e^{a\theta t}(-V(0, y(0)))$$

From this and from inequalities (2.40) and (2.41) we have the inequality

$$-\omega h(t) y_n(t)^2 \geq e^{a\theta t}(-V(0, y(0))), \quad \forall t \geq 0$$

Thus

$$y_n(t)^2 \geq \frac{e^{a\theta t}}{\omega \sup_t(-h(t))}(-V(0, y(0))) \tag{2.45}$$

The Lyapunov instability of the solution $y(t) \equiv 0$ follows from this inequality. Moreover, from estimate (2.45) it follows that, in the neighbourhood of $y = 0$, the solution $y(t)$ with initial data $V(0, y(0)) < 0$ increases exponentially.

Since $d > 0$ and $U(t)$ is a unitary matrix, the zero solution of system (2.1) is also Lyapunov unstable.

Note that the requirement of uniformity with respect to τ in estimates (2.30) and (2.31) is also characteristic of the first-approximation stability criteria [11, 16, 17].

The problem of weakening the instability conditions obtained in Theorems 1 and 2 naturally arises. However, the Perron effects described in Section 1 set limits to such weakening.

This research was partially supported by the Russian Foundation for Basic Research (01-01-00317), the "Universities of Russia" programme and the "State Support for Leading Scientific Schools" programme (NSh-2257.2003.01)

REFERENCES

1. MAXWELL, J. C., VYSHNEGRADSKII, I. A. and STODOLA, A. *Automatic Control Theory. Linearized Problems*. Izd. Akad. Nauk SSSR, Moscow, 1949.
2. NEIMARK, Yu. I. and LANDA, P. S., *Stochastic and Chaotic Vibrations*. Nauka, Moscow, 1987.
3. MOON, F., *Chaotic Vibrations*. John Wiley. New York, 1987.
4. LANDA, P. S., *Nonlinear Oscillations and Waves in Dynamical Systems*. Kluwer, Dordrecht.
5. BERGE, P., POMEAU, Y. and VIDAL, C., *L'Ordre dans le Chaos*. Hermann, Paris, 1984.
6. RYABOV, V. B., Using Lyapunov exponents to predict the onset of chaos in nonlinear oscillator. *Phys. Rev. E*, 2002, **66**, 1, 16 214–16 231.
7. RUSSEL, D. A., HANSON, J. D. and OTT, E., Dimension of strange attractors. *Phys. Rev. Lett.*, 1980, **45**, 14, 1175–1178.
8. LEONOV, G. A., Formulae of Lyapunov dimension of Xenon and Lorentz attractors. *Algebra i Analiz*, 2001, **13**, 3, 155–170.
9. LEONOV, G. A., Strange attractors and the classical theory of the stability of motion. *Uspekhi Mekhaniki*, 2002, **1**, 3, 3–42.
10. PERRON, O., Die Stabilitätsfrage bei Differentialgleichungen. *Math. Z.*, 1930, **32**, 5, 703–728.
11. MALKIN, I.G., *Theory of the Stability of Motion*. Nauka, Moscow, 1966.
12. FILIPPOV, A. F., *Differential Equations with a Discontinuous Right-hand Side*. Nauka, Moscow, 1985.
13. DEMIDOVICH, B. P., *Lectures on the Mathematical Theory of Stability*. Nauka, Moscow, 1967.
14. CHETAYEV, N. G., *The Stability of Motion*. Gostekhizdat, Moscow, 1955.
15. CHETAYEV, N. G., Some problems of the stability and instability for incorrect systems. *Prikl. Mat. Mekh.*, 1948, **12**, 6, 639–642.
16. BYLOV, B. F., VINOGRAD, R. E., GROBMAN, D. M. and NEMYTSKII, V. V., *The Theory of Lyapunov Exponents and its Application to Stability Problems*. Nauka, Moscow, 1966.
17. DALETSKII, Yu. L. and KREIN, M. G., *The stability of Solutions of Differential Equations in Banach Space*. Nauka, Moscow, 1970.

Translated by P.S.C.